On linear operators with an invariant subspace of functions

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Abstract

Let us denote $\mathcal{V}$, the finite dimensional vector spaces of functions of the form $\psi(x) = p_n(x) + f(x)p_m(x)$ where $p_n(x)$ and $p_m(x)$ are arbitrary polynomials of degree at most n and m in the variable x while $f(x)$ represents a fixed function of x. Conditions on m, n and $f(x)$ are found such that families of linear differential operators exist which preserve $\mathcal{V}$. A special emphasis is accorded to the cases where the set of differential operators represents the envelopping algebra of some abstract algebra.

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1 Introduction

Quasi Exactly Solvable (QES) operators are characterized by linear differential operators which preserve a finite-dimensional vector space $\mathcal{V}$ of smooth functions [1]. In the case of operators of one real variable the underlying vector space is often of the form $\mathcal{V} = \mathcal{P}_n$ where $\mathcal{P}_n$ represents the vector space of polynomials of degree at most n in the variable x. In [2] it is shown that the linear operators preserving $\mathcal{P}_n$ are generated by three basic operators j_-, j_0, j_+ (see Eq.(1) below) which realize the algebra $sl(2, \mathbb{R})$. More general QES operators can then be constructed by considering the elements of the enveloping algebra of these generators, performing a change of variable and/or conjugating the j's with an invertible function, say $g(x)$. The effective invariant space is then the set of functions of the form $g(x)\mathcal{P}_n$.

In this paper we consider a more general situation. Let m, n be two positive integers and let $f(x)$ be a sufficiently derivable function in a domain of the real line. Let $\mathcal{V} = \mathcal{P}_n + f(x)\mathcal{P}_m$ be the vector space of functions of the form $p(x) + f(x)q(x)$ where $p(x) \in \mathcal{P}_n$, $q(x) \in \mathcal{P}_m$.

We want to address the following questions : What are the differential operators which preserve $\mathcal{V}$? and for which choice of $m, n, f(x)$ do these operators posses a relation with the enveloping algebra of some Lie (or "deformed" Lie) algebra ?. This question generalizes the cases of monomials adressed in [3] and more recently in [4] At the moment the question is, to our knowledge, not solved in its generality but we present a few non trivial solutions in the next section.

2 Examples

2.1 Case $f(x) = 0$

This is off course the well known case of [1, 2]. The relevant operators read

$$j_+(n) = x(x\frac{d}{dx} - n) \quad , \quad j_0(n) = (x\frac{d}{dx} - \frac{n}{2}) \quad , \quad j_- = \frac{d}{dx} \tag{1}$$

and represent the three generators of $sl(2, \mathbb{R})$. Most of known one-dimensional QES equations are build with these operators. For later convenience, we further define a family of equivalent realizations of $sl(2, \mathbb{R})$ by means of the conjugated operators $k_\epsilon(a) \equiv x^a j_\epsilon x^{-a}$ for $\epsilon = +, 0, -$ and a is a real number.

2.2 Case $f(x) = x^a$

The general cases of vector spaces constructed over monomials was first adressed in [3] and the particular subcase $f(x) = x^a$ was reconsidered recently [4]. The corresponding vector space was denoted $V^{(1)}$ in [4]; here we will reconsider this case and extend the discussion of the operators which leave it invariant. For later convenience, it is usefull

to introduce more precise notations, setting $\mathcal{P}_n \equiv \mathcal{P}(n,x)$ and

$$\begin{aligned} V^{(1)} \equiv V^{(1)}(N,s,a,x) &= \mathcal{P}(n,x) + x^a \mathcal{P}(m,x) \\ &= \mathrm{span}\{1, x, x^2, \ldots, x^n; x^a, x^{a+1}, \ldots, x^{a+m}\} \\ &= V_1^{(1)} \oplus V_2^{(1)} \end{aligned} \tag{2}$$

in passing, note that the notations of [4] are $n = s$ and $m = N - s - 2$.

The vector space above is clearly constructed as the direct sum of two subspaces. As pointed out in [3, 4] three independent, second order differential operators can be constructed which preserve the vector space $V^{(1)}$. Writing these operators in the form

$$\begin{aligned} J_+ &= x(x\frac{d}{dx} - n)(x\frac{d}{dx} - (m+a)) \\ J_0 &= (x\frac{d}{dx} - \frac{m+n+1}{2}) \\ J_- &= (x\frac{d}{dx} + 1 - a)\frac{d}{dx} \end{aligned} \tag{3}$$

makes it obvious that they preserve $V^{(1)}$.

These operators close under the commutator into a polynomial deformation of the $sl(2,\mathbb{R})$ algebra:

$$\begin{aligned} [J_0, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= \alpha J_0^3 + \beta J_0^2 + \gamma J_0 + \delta \end{aligned} \tag{4}$$

where $\alpha, \beta, \gamma, \delta$ are constants given in [4].

Clearly the operators (3) leave separately invariant two vector spaces $V_1^{(1)}$ and $V_1^{(1)}$ entering in (2). In the language of representations they act reducibly on $V^{(1)}$. However, operators can be constructed which preserve $V^{(1)}$ while mixing the two subspaces. The form of these supplementary operators is different according to the fact that the number a is an integer or not; we now adress these two cases separately.

2.2.1 $a \in \mathbb{R}$

In order to construct the operators which mix $V_1^{(1)}$ and $V_2^{(1)}$, we first define

$$\begin{aligned} K &= (D-n)(D-n+1)\ldots D \quad , \quad D \equiv x\frac{d}{dx} \\ K' &= (D-m-a)(D-m-a+1)\ldots(D-a) \end{aligned} \tag{5}$$

which belong to the kernals of the subvector spaces $\mathcal{P}_n$ and $x^a\mathcal{P}_m$ of $V^{(1)}$ respectively. Notice that the products $j_\epsilon \tilde{K}$ and $k_\epsilon(a)K$ (with $\epsilon = 0, \pm$) also preserve the vector space. For generic values of m, n these operators contain more than second derivatives and, as so, they were not considered in [4].

In order to construct the operators which mix the two vector subspaces entering in $\mathcal{V}$, we first have to construct the operators which transform a generic element of

$\mathcal{P}_m$ into an element of $\mathcal{P}_n$ and vice-versa. In [5] it is shown that these operators are of the form

$$
\begin{aligned}
q_\alpha &= x^\alpha \quad , \quad \alpha = 0, 1, \ldots, \Delta && (6)\\
\overline{q}_\alpha &= \prod_{j=0}^{\alpha-1} (D - (p+1-\Delta) - j)(\frac{d}{dx})^{\Delta-\alpha} && (7)
\end{aligned}
$$

where $\Delta \equiv |m-n|$, $p \equiv$max$\{m, n\}$

The operators preserving $\mathcal{V}$ while exchanging the two subspaces can finally be constructed by means of

$$
Q_\alpha = q_\alpha x^{-a} K \quad , \quad \overline{Q}_\alpha = x^a \overline{q}_\alpha K' \quad , \quad \alpha = 0, 1, \ldots, \Delta. \tag{8}
$$

Here we assumed $n \leq m$, the case $n \geq m$ is obtained by exchanging q_α with $\overline{q}_\alpha$ in the formula above.

It can be checked easily that Q_α, transform a vector of the form $p_n + x^a q_m$ into a vector of the form $\tilde{q}_n \in \mathcal{P}_n$ while $\overline{Q}_\alpha$ transforms the same vector into a vector of the form $x^a \tilde{p}_m \in x^a \mathcal{P}_m$.

The generators constructed above are in one to one correspondance with the 2×2 matrix generators preserving the direct sum of vector spaces $\mathcal{P}_m \oplus \mathcal{P}_n$ classified in [5] although their form is quite different (the same notation is nevertheless used). The commutation relations (defining a normal order) which the generators fullfill is also drastically different as we shall discuss now. First of all it can be easily checked that all products of operators Q (and separately of $\overline{Q}$) belong to the kernal of the full space $V^{(1)}$, so we can write

$$
Q_\alpha Q_\beta = \overline{Q}_\alpha \overline{Q}_\beta = 0 \tag{9}
$$

which suggests that the operators Q's and the $\overline{Q}$'s play the role of fermionic generators, in contrast to the J's which are bosonic (note that the same distinction holds in the case [5]).

From now on, we assume $n = m$ in this section (the evaluation of the commutators for generic values of m, n is straightforward but leads to even more involved expressions) and suppress the superflous index α on the the fermionic operators. The commutation relations between fermionic and bosonic generators leads to

$$
\begin{aligned}
[Q, J_-] &= (2a - n - 1) j_- Q \ , \qquad [Q, J_+] = (2a + n + 1) j_+ Q \\
[\overline{Q}, J_-] &= -(2a + n + 1) k_-(a) \overline{Q} \ , \quad [\overline{Q}, J_+] = -(2a - n - 1) k_+(a) \overline{Q}
\end{aligned} \tag{10}
$$

where the $j_\pm$ and $k_\pm(a)$ are defined in (1). These relations define a normal order but we notice that the right hand side are not linear expressions of the generators choosen as basic elements. We also have

$$
[Q, D] = (D + a) Q \quad , \quad [\overline{Q}, D] = (D - a) \overline{Q} \tag{11}
$$

This is to be contrasted with the problem studied in [5] where, for the case $\Delta = 0$, the Q (and the $\overline{Q}$) commute with the three bosonic generators, forming finally an sl(2)×

sl(2) algebra. Here we see that the bosonic operators J and fermionic operators $Q, \overline{Q}$ do not close linearly under the commutator. The commutators involve in fact extra factors which can be expressed in terms of the operators j or $k(a)$ acting on the appropriate subspace $\mathcal{P}_m$. This defines a normal order among the basic generators but makes the underlying algebraic stucture (if any) non linear. For completeness, we also mention that the anti-commutator $\{Q, \overline{Q}\}$ is a polynomial in J_0.

2.2.2 $a \in \mathbb{N}$

Let us consider the case $a \equiv k \in \mathbb{N}_0$, with $n \leq k$ and assume for definiteness $m - k \geq n$. Operators that preserve $V^{(1)}$ while exchanging some monomials of the subspace $V_1^{(1)}$ with some of $V_2^{(1)}$ (and vice versa) can be expressed as follows :

$$W_+ = x^k \prod_{j=0}^{k-1} (D - k - m + j) \tag{12}$$

$$W_- = \frac{1}{x^k} \prod_{j=0}^{n} (D - j) \prod_{i=1}^{k-n-1} (D - k - n - i) \tag{13}$$

These operators are both of order k, W_+ is of degree k while W_- is of degree $-k$. When acting on the monomial of Eq.(2), W_+ transforms the $n+1$ monomials of $V_1^{(1)}$ into the first $n+1$ monomials of $V_2^{(1)}$ and annihilates the k monomials of highest degrees in $V_2^{(1)}$. To the contrary W_- annihilates the $n+1$ monomials of $V_1^{(1)}$ and shifts the $n+1$ monomials of lowest degrees of $V_2^{(1)}$ into $V_1^{(1)}$. Operators of the same type performing higher jumps can be constructed in a straighforward way; they are characterized by a higher order and higher degrees but we will not present them here.

The two particular cases $k = n + 1$ and $k = 2$ can be further commented. In the case $k = n + 1$ the space $V^{(1)}$ is just $\mathcal{P}_{m+n+1}$ and the operators W_+, W_- can be rewritten as

$$W_+ = (j_+(m + n + 1))^{n+1} \quad , \quad W_- = (j_-)^{n+1} \tag{14}$$

where $j_\pm$ are defined in (1). Setting $k = 2$ (and $n = 0$ otherwise we fall on the case just mentionned), we see that the operators $W_\pm$ become second order and coincide with the operators noted $T_2^{(+2)}$, $T_2^{(-2)}$ in Sect. 4 of the recent preprint [10]. With our notation they read

$$W_+ = x^2(D - (m + 2))(D - (m + 1)) \quad , \quad W_- = x^{-2}D(D - 3) \tag{15}$$

A natural question which come out is to study whether the non-linear algebra (4) is extended in a nice way by the supplementary operators $W_\pm$ and their higher order counterparts. So far, we have not found any interesting extended structure. For example, for $k = 2, n = 0$, we computed :

$$\begin{aligned}
[W_+, J_+] &= -2x^3(D - (m + 2))(D - (m + 1))(D - m) \\
[W_+, J_-] &= -6xD(D - (m + 2))(D - \frac{2}{3}(m + 2))
\end{aligned} \tag{16}$$

which just show that the commutators close within the envelopping algebra of the $V^{(1)}$ preserving operators but would need more investigation to be confirmed as an abstract algebraic stucture.

We end up this section by mentionning that the two other vector spaces constructed in [4] and denoted $V^{(a-1)}$ and $V^{(a)}$ can in fact be related to $V^{(1)}$ by means of the following relations :

$$V^{(a-1)}(x) = V^{(1)}(N, s = 0, \frac{1}{a-1}, x^{a-1}) \tag{17}$$

$$V^{(a)}(x) = V^{(1)}(N, s, \frac{1}{a}, x^{a}) \tag{18}$$

Off course the operators preserving them can be obtained from the operators above (3) after a suitable change of variable and the results above can easily be extended to these vector spaces.

2.3 Case $f(x) = \sqrt{p_2(x)}$, $m = n - 1$

Here, $p_2(x)$ denotes a polynomial of degree 2 in x, we take it in the canonical form $p_2(x) = (1-x)(1-\lambda x)$. In the case $m = n-1$, three basic operators can be contructed which preserve $\mathcal{V}$; they are of the form the form

$$\begin{aligned} S_1 &= nx + p_2 \frac{d}{dx} \\ S_2 &= \sqrt{p_2}(nx - x\frac{d}{dx}) \\ S_3 &= \sqrt{p_2}(\frac{d}{dx}) \end{aligned} \tag{19}$$

and obey the commutation relations of so(3). The family of operators preserving $\mathcal{V}$ is in this case the enveloping algebra of the Lie algebra of SO(3) in the realization above. Two particular cases are worth to be pointed out :

- $\lambda = -1$
  Using the variable $x = \cos(\phi)$, the vector space $\mathcal{V}$ can be re-expressed is the form
  $$\mathcal{V} = \text{span}\{\cos(n\phi), \sin(n\phi), \cos((n-1)\phi), \sin((n-1)\phi), \ldots\} \tag{20}$$
  and the operators S_a above can be expressed in terms of trigonometric functions. Exemples of QES equations of this type were studied in [6] in relation with spin systems.

- $\lambda = k^2$ Using the variable $x = \text{sn}(z, k)$, (with $\text{sn}(z, k)$ denoting the Jacobi elliptic function of modulus k, $0 \leq k^2 \leq 1$), and considering the Lamé equation :

$$-\frac{d^2\psi}{dz^2} + N(N+1)k^2 \text{sn}^2(z, k)\psi = E\psi \tag{21}$$

It in known (see e.g. [7]) that doubly periodic solutions exist if N is a semi integer. If $N = (2n+1)/2$ these solutions are of the form

$$\psi(z) = \sqrt{\mathrm{cn}(z,k) + \mathrm{dn}(z,k)}(p_n(x) + \mathrm{cn}(z,k)\mathrm{dn}(z,k)p_{n-1}(x)) \tag{22}$$

where $\mathrm{cn}(z,k), \mathrm{dn(z,k)}$ denote the other Jacobi elliptic functions. The second factor of this expression is exactly an element of the vector space under consideration. The relations between the doubly periodic solutions of the Lamé equation and QES operators was pointed out in [8].

2.4 Case $f(x) = \sqrt{(1-x)/(1-\lambda x)}$, $m = n$

In this case again, the vector space $\mathcal{V}$ for $m = n$ is preserved by the three operators S_a of Eq.(19) provided $m = n$. Two cases are worth considering, in complete paralelism with Sect. 2.3:

- $\lambda = -1$. Using the new variable $x = \cos\phi$, and using the identity $\tan(\phi/2) = \sqrt{(1-x)/(1+x)}$ the vector space $\mathcal{V}$ can be reexpressed is the form

$$\mathcal{V} = \mathrm{span}\{\cos(\frac{2n+1}{2}\phi), \sin(\frac{2n+1}{2}\phi), \cos(\frac{2n-1}{2}\phi), \sin(\frac{2n-1}{2}\phi), \ldots\} \tag{23}$$

  and exemples of QES operators having solutions in this vector space are presented in [6].

- $\lambda = k^2$. Again, in this case, the variable $x = \mathrm{sn}(z,k)$ is usefull and the doubly periodic solutions of the Lamé equation (22) corresponding to $N = (2n+3)/2$ of the form [8]

$$\psi(z) = \sqrt{\mathrm{cn}(z,k) + \mathrm{dn}(z,k)}(\mathrm{cn}(z,k)p_n(x) + \mathrm{dn}(z,k)q_n(x)) \tag{24}$$

  provide examples of QES solutions constructed in the space under consideration.

3 Conclusions

The problem considered in this paper enlarges the framework where QES operators are usually constructed and offers a variety of non trivial forms of the function $f(x)$. Very recently, similar kinds of extensions of QES operators were found [10], although treated with a different orientation than the one proposed here.

Several QES operators obtained in different contexts [6, 8, 4] are recovered by our method in a unified way. Other attempts with different form of the function $f(x)$ turned out to be trivial or to admit very involved (or very poor) sets of preserving operators. Other simple forms of the function $f(x)$ could definitely be looked for. The construction of QES operators preserving the space $\mathcal{V} = \mathcal{P}_n + f\mathcal{P}_m$ can be generalized in many directions. Namely (i) to functions of several variables, (ii) to the matrix case

(i.e to a space defined by the direct sum two or more spaces of the type $\mathcal{V}$). Apart from the algebraic problem of classifying the linear differential operators preserving these vector spaces, the construction of new QES Schrödinger operators constitutes another interesting problem. The quantum hamiltonians constructed in [9] are constructed along the line (ii) mentionned above.

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